Dimensions of Invariant Secant Varieties

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Additive rank decompositions

We study problems, which have the following form: Given a low-dimensional variety X in a high-dimensional space $\mathcal{L}$ and $f \in \mathcal{L}$, find $x_1, \ldots, x_m \in X$ such that

$$f = x_1 + \ldots + x_m. \quad (1)$$

X is assumed to be irreducible and nondegenerate.

A surprising lot of problems fall into this category:

1. The rank decomposition of a matrix f is obtained for $X = \{uv^T \mid u, v \in \mathbb{C}^n\}$. Indeed, a matrix of rank at most m decomposes as $f = u_1 v_1^T + \ldots + u_m v_m^T$.
2. The rank decomposition of a symmetric tensor is obtained, if $X = \{v^{\otimes d} \mid v \in \mathbb{C}^n\}$ is taken as the Veronese variety. Indeed, a symmetric tensor f of rank at most m decomposes as $f = v_1^{\otimes d} + \ldots + v_m^{\otimes d}$.
3. Skew-symmetric tensor decomposition is obtained, if $X = \{v_1 \wedge \ldots \wedge v_d \mid v_1, \ldots, v_d \in \mathbb{C}^n\}$ is taken as the (affine cone of the) Grassmannian variety $\mathrm{Gr}(d, n)$. Indeed, a skew symmetric tensor f of skew rank m decomposes as $f = (v_{11} \wedge \ldots \wedge v_{1d}) + \ldots + (v_{m1} \wedge \ldots \wedge v_{md})$.

Questions

Q1: Generic rank: What is the minimal $m = m_g$ needed to represent a general $f \in \mathcal{L}$?

Q2: Identifiability: Is the decomposition unique, when m is small and f is built from general $x_1, \ldots, x_m \in X$? What is the largest $m = m_{\mathrm{id}}$ such that generic m-fold sum decompositions are unique?

Q2': Nondefectivity: What is the maximum $m = m_0$, such that a general sum of m elements of X has finitely many decompositions as m-fold sums from X?

Note that

$$m_0 - 1 \overset{(\text{if } X \text{ smooth})}{\leq} m_{\mathrm{id}} \leq m_0 \leq \frac{\dim \mathcal{L}}{\dim X} \leq m_g \quad (2)$$

The first inequality is a very recent result, shown in [5]. As a consequence, Q2 and Q2' often have almost the same answer. We want to understand the gap between m_0 and m_g. We give the following partial answer:

Answer: $m_g - m_0 \leq \dim X - 1$ (if X satisfies some conditions). (3)

Overview

1. The **generic rank** is trivially known to be $m_g = n$ for $n \times n$ matrices. Since matrices of rank-2 have infinitely many rank decompositions, it holds $m_0 = 1$. Therefore, the bound $m_g - m_0 = \dim X - 1$ is tight.
2. The generic rank of Waring decompositions in degree at least 3 is known by the famous Alexander-Hirschowitz theorem. Here, it always holds that $m_g - m_0 \leq 2$. There are only 5 exceptional cases where the difference is two. In all other cases, $m_g - m_0 \leq 1$.
3. The generic rank of skew-symmetric tensor decomposition is unknown. A solution would have importance to coding theory.
4. The generic rank of partially symmetric tensors (Segre-Veronese varieties) has recently been determined for larger degrees, see [1]. Abo et al. used our bound on $m_g - m_0$ to obtain starting cases for their induction proof. Figure 2 shows a case that is still open.

Examples

The tensor $f = e_1 \otimes e_2 \otimes e_3^{\otimes 2} + e_4 \otimes e_5 \otimes e_6^{\otimes 2} \in (\mathbb{C}^6)^{\otimes 2} \otimes S^2(\mathbb{C}^6)$ has a unique representation as a sum of at most two elements of $X = \{v_1 \otimes v_2 \otimes v_3^{\otimes 2} \mid v_1, v_2, v_3 \in \mathbb{C}^6\}$.

A general tensor $f \in (\mathbb{C}^6)^{\otimes 2} \otimes S^2(\mathbb{C}^6)$ can be written as a sum of at most 63 elements from X. Generic tensors of X-rank at most 32 are identifiable.

Theorem (B. and Casarotti, 2023, [2])

Let G a group and let X be an irreducible affine cone living in an irreducible G-module $\mathcal{L}$. Assume X is G-invariant (thus $h \cdot X = X$ for all $h \in G$). Then, it holds

$$m_g - m_0 \leq \dim X - 1. \quad (4)$$

In particular, $m_g \leq \frac{\dim \mathcal{L}}{\dim X} + \dim X - 1$ and $m_0 \geq \frac{\dim \mathcal{L}}{\dim X} - \dim X + 1$.

Terracini's Lemma

For an irreducible affine cone X, we define the m-th **secant variety** as the (Zariski) closure of the set of m-fold sums of X:

$$\sigma_m(X) = \overline{\{x_1 + \ldots + x_m \mid x_i \in X\}} \quad (5)$$

Terracini's Lemma says: For general $x_1, \ldots, x_m \in X$ and general $f \in \langle x_1, \ldots, x_m \rangle$, the tangent space to $\sigma_m(X)$ at f equals $T_{x_1}X + \ldots + T_{x_m}X$.

Therefore, the **generic rank** is the least $m_g \in \mathbb{N}$ such that a sum of m_g general tangent spaces to X equals $\mathcal{L}$.

For general $f \in \sigma_m(X)$, the decomposition problem $f = x_1 + \ldots + x_m$ has **finitely many** solutions $\{x_1, \ldots, x_m\}$ with $x_i \in X$, if and only if the sum of tangent spaces $T_{x_1}V_1 + \ldots + T_{x_m}V_m$ at general $x_1, \ldots, x_m \in X$ is a direct sum.

Example Applications

1. For Waring decompositions, take $X = \{v^{\otimes d} \mid v \in \mathbb{C}^n\}$ as the **Veronese** variety, $\mathcal{L} = S^d(\mathbb{C}^n)$ as the space of symmetric d-tensors on $\mathbb{C}^n$ and $G = \mathrm{GL}(\mathbb{C}^n)$.
2. For skew-symmetric decompositions, take $X = \mathrm{Gr}(d, n)$ as the affine **Grassmannian**, $\mathcal{L} = \bigwedge^d(\mathbb{C}^n)$ as the space of alternating d-tensors on $\mathbb{C}^n$ and $G = \mathrm{GL}(\mathbb{C}^n)$.
3. For standard tensor decompositions, take $X = \mathrm{Segre}(n_1, \ldots, n_d) = \{v_1 \otimes \ldots \otimes v_d \mid v_i \in \mathbb{C}^{n_i}\}$ as the **Segre** variety, $\mathcal{L} = \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$ and $G = \mathrm{GL}(\mathbb{C}^{n_1}) \times \ldots \times \mathrm{GL}(\mathbb{C}^{n_d})$.
4. **Segre-Veronese** varieties $X = \mathrm{SegVer}(n_1^{d_1}, \ldots, n_t^{d_t}) = \{v_1^{\otimes d_1} \otimes \ldots \otimes v_t^{\otimes d_t} \mid v_i \in \mathbb{C}^{n_i}\}$ live in $\mathcal{L} = S^{d_1}(\mathbb{C}^{n_1}) \otimes \ldots \otimes S^{d_t}(\mathbb{C}^{n_t})$. They capture tensor decompositions with partial symmetries. Here, the group is $G = \mathrm{GL}(\mathbb{C}^{n_1}) \times \ldots \times \mathrm{GL}(\mathbb{C}^{n_t})$.

The Grassmannian $\mathrm{Gr}(5, n)$ in $\bigwedge^5(\mathbb{C}^n)$

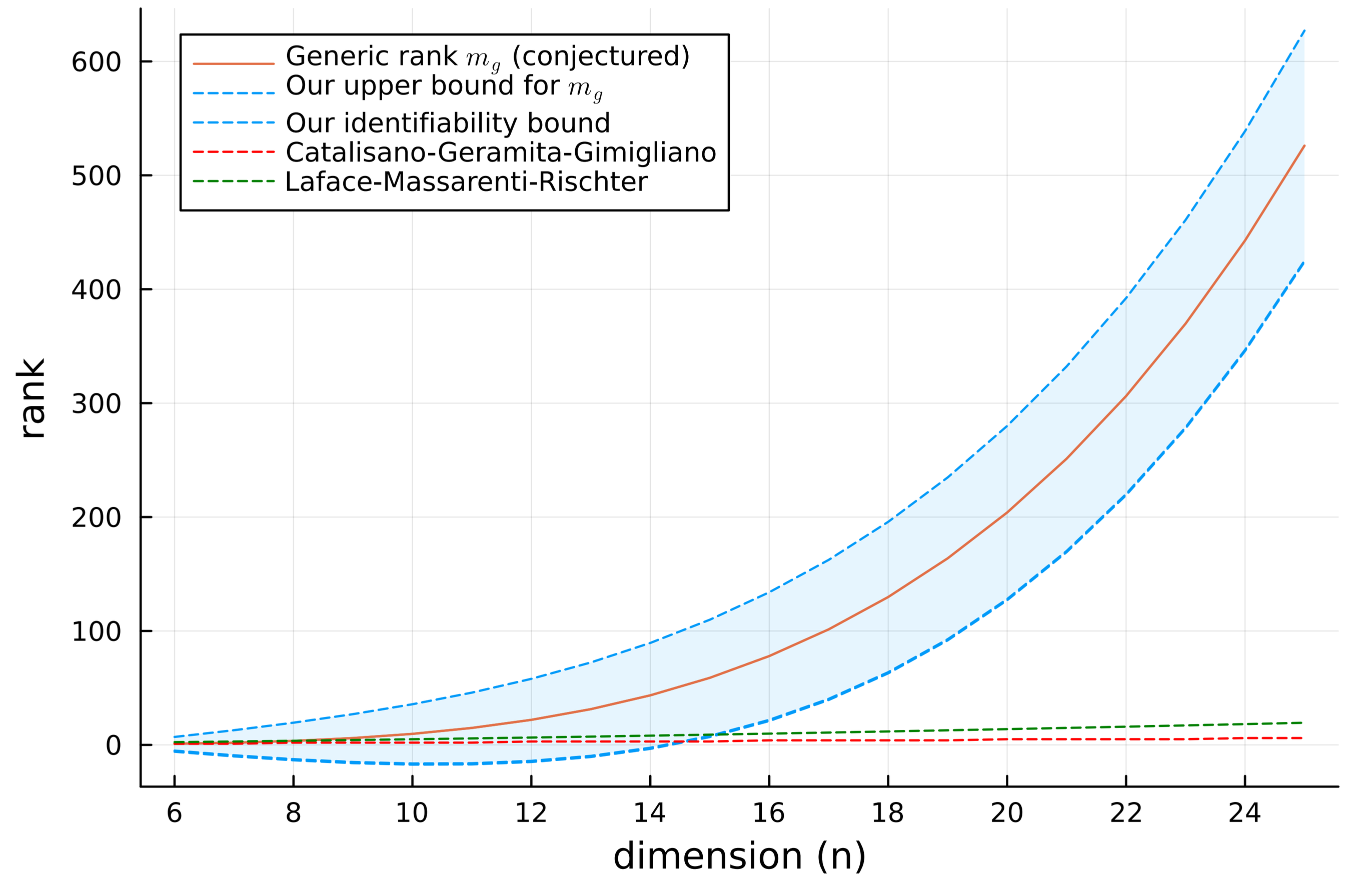

Figure 1. **The Grassmannian variety** $X = \{v_1 \wedge \ldots \wedge v_5 \mid v_i \in \mathbb{C}^n\}$: Comparison of our bound on nondefectivity (lower blue curve) with previously best-known bound on m_0 by Laface-Massarenti-Rischter [4] and Catalisano-Geramita-Gimigliano [3]. The upper blue curve shows our upper bound for the generic rank m_g.

Proof sketch (for brave & caffeinated readers)

There is a largest number m_0 such that a sum $T_{x_1}X + \ldots + T_{x_{m_0}}X$ of m_0 general tangent spaces to X is direct. Here, $x_1, \ldots, x_{m_0}$ denote general points in X.

There is a smallest number m_g such that $T_{x_1}X + \ldots + T_{x_{m_g}}X = \mathcal{L}$ for any general choice of $x_1, \ldots, x_{m_g}$. This number m_g coincides with the generic rank.

We claim that $m_g - m_0 \leq \dim X - 1$. To see this, consider general $x_1, x_2, \ldots \in X$ and general $y \in X$. Define the sequence $a_1 \leq a_2 \leq a_3 \leq \ldots$ via

$$a_m := \dim(T_{x_1}X + \ldots + T_{x_m}X) \cap T_yX \quad (6)$$

Note that a_{m_0} is the first nonzero value of the sequence and $a_{m_g} = \dim X$. We claim that the sequence is strictly monotone between m_0 and m_g. Indeed, if this is true, then

$$m_g - m_0 \leq a_{m_g} - a_{m_0} \leq \dim X - 1. \quad (7)$$

To the contrary, assume there was a stationarity point $a_m = a_{m+1}$ of the sequence with $a_m > 0$. Then for all general $z \in X$, it holds that

$$U_y := (T_{x_1}X + \ldots + T_{x_m}X) \cap T_yX = (T_{x_1}X + \ldots + T_{x_m}X + T_zX) \cap T_yX, \quad (8)$$

since the left-hand-side is a subspace of the RHS and both have the same dimension a_m. This means we can introduce any general point $z \in X$ without changing the space U_y. By the same reasoning backwards, we can in fact **exchange** z with any of $x_1, \ldots, x_m$, without changing the space U_y. Therefore, $U_y = (T_{x_1}X + \ldots + T_{x_{m-1}}X + T_zX) \cap T_yX$.

Applying this inductively, we can exchange $x_1, \ldots, x_m$ by any set $z_1, \ldots, z_m$ of general points of X. Hence $U_y = (T_{z_1}X + \ldots + T_{z_m}X) \cap T_yX$ for all general z_i. Now, choose $z_i = gx_i$ for some $g \in G$. The previous argument shows $gU_y = U_{gy}$. In other words, the space $T_{x_1}X + \ldots + T_{x_m}X$ contains the subspace gU_y for any $g \in G$. Therefore, it contains the G-module generated by U_y. Since $\dim U_y = a_m > 0$, this G-module cannot be zero. Since the G-module $\mathcal{L}$ is irreducible, we conclude that $T_{x_1}X + \ldots + T_{x_m}X = \mathcal{L}$.

The Segre-Veronese variety in $(\mathbb{C}^n)^{\otimes 2} \otimes S^2(\mathbb{C}^n)$

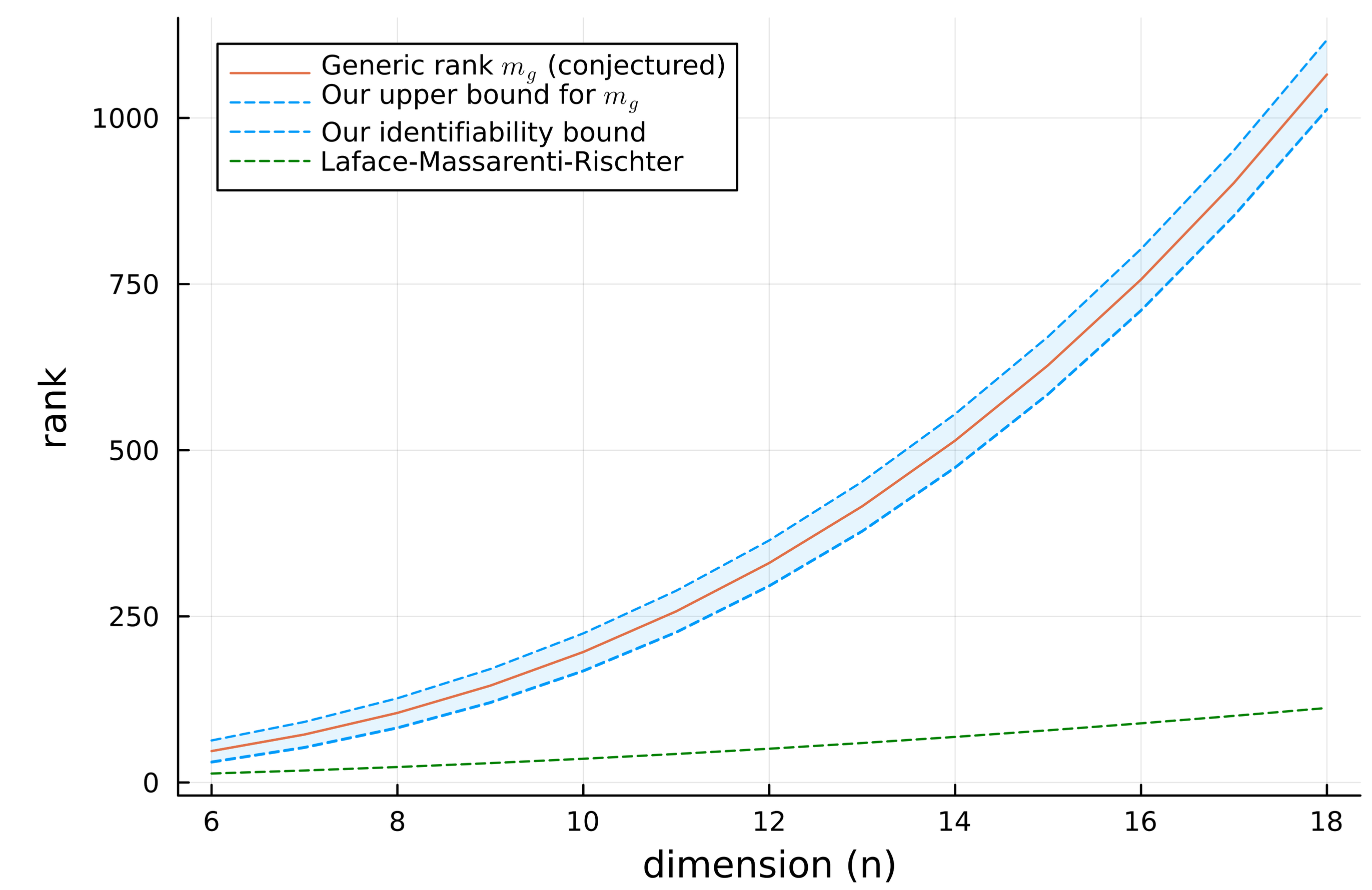

Figure 2. **The Segre-Veronese variety** $X = \{v_1 \otimes v_2 \otimes v_3^{\otimes 2} \mid v_1, v_2, v_3 \in \mathbb{C}^n\}$: Comparison of our bounds for m_0 and m_g with the previous best-known bound for m_0 by Laface-Massarenti-Rischter.

References

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